

# On the Random Conjugate Spaces of a Random Locally Convex Module \*

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**Abstract** Theoretically speaking, there are four kinds of possibilities to define the random conjugate space of a random locally convex module. The purpose of this paper is to prove that among the four kinds there are only two which are universally suitable for the current development of the theory of random conjugate spaces: in this process we also obtain a somewhat surprising and crucial result that for a random normed module with base  $(\Omega, \mathcal{F}, P)$  such that  $(\Omega, \mathcal{F}, P)$  is nonatomic then the random normed module is a totally disconnected topological space when it is endowed with the locally  $L^0$ -convex topology.

**Keywords** Random locally convex modules,  $(\varepsilon, \lambda)$ -topology, locally  $L^0$ -convex topology, random conjugate spaces

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## 1 Introduction and main results

The notion of a random locally convex module is a random generalization of that of a locally convex space. However, the theory of classical conjugate spaces for locally convex spaces universally fails to serve for the development of random locally convex modules, which motivated us to have developed the theory of random conjugate spaces. Now, the theory of random conjugate spaces has played an essential role in both the theory of random locally convex modules and the theory of conditional risk measures, see [1, 2] for details.

Since there are two kinds of useful topologies for every random locally convex module, namely the  $(\varepsilon, \lambda)$ -topology introduced by Guo in [3] and the locally  $L^0$ -convex topology introduced by Filipović, et.al in [4], we can naturally consider four kinds of possibilities to define the random conjugate space of a random locally convex module. The purpose of this paper is to further discuss the relations among the four kinds of definitions.

To introduce the main results of this paper, let us first recall some notation and terminology as follows.

Throughout this paper,  $(\Omega, \mathcal{F}, P)$  denotes a probability space,  $K$  the real number field  $R$  or the complex number field  $C$  and  $L^0(\mathcal{F}, K)$  the algebra of equivalence classes of  $K$ -valued measurable random variables on  $\Omega$  under the ordinary scalar multiplication, addition and multiplication operations on equivalence classes.

Given a random locally convex module  $(E, \mathcal{P})$  over  $K$  with base  $(\Omega, \mathcal{F}, P)$ , let  $\mathcal{T}_{\varepsilon, \lambda}$  and  $\mathcal{T}_c$  denote the  $(\varepsilon, \lambda)$ -topology and the locally  $L^0$ -convex topology for  $E$ , respectively, see

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[1, 4] and also Section 2 for the definitions of these two kinds of topologies.  $L^0(\mathcal{F}, K)$ , as a special random locally convex module, also has the corresponding two kinds of topologies. The  $(\varepsilon, \lambda)$ -topology for  $L^0(\mathcal{F}, K)$  is exactly the topology of convergence in probability  $P$ ,  $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})$  is a topological algebra over  $K$ , and in particular when  $(E, \mathcal{P})$  is endowed with the  $(\varepsilon, \lambda)$ -topology  $(E, \mathcal{P})$  is a topological module over the topological algebra  $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})$ . However,  $(L^0(\mathcal{F}, K), \mathcal{T}_c)$  is only a topological ring, namely the locally  $L^0$ -convex topology for  $L^0(\mathcal{F}, K)$  is not necessarily a linear topology (see [4] for details), and in particular when  $(E, \mathcal{P})$  is endowed with the locally  $L^0$ -convex topology  $(E, \mathcal{P})$  is a topological module over the topological ring  $(L^0(\mathcal{F}, K), \mathcal{T}_c)$ .

We can now introduce the following:

**Definition 1.1** Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$  and define  $E_{\varepsilon, \lambda}^*$ ,  $E_c^*$ ,  $E_{max}^*$  and  $E_{min}^*$  as follows:

- (1)  $E_{\varepsilon, \lambda}^* = \{f \mid f \text{ is a continuous module homomorphism from } (E, \mathcal{T}_{\varepsilon, \lambda}) \text{ to } (L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})\}$ ,
- (2)  $E_c^* = \{f \mid f \text{ is a continuous module homomorphism from } (E, \mathcal{T}_c) \text{ to } (L^0(\mathcal{F}, K), \mathcal{T}_c)\}$ ,
- (3)  $E_{max}^* = \{f \mid f \text{ is a continuous module homomorphism from } (E, \mathcal{T}_c) \text{ to } (L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})\}$ ,

where  $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})$  is regarded as a topological ring;

- (4)  $E_{min}^* = \{f \mid f \text{ is a continuous module homomorphism from } (E, \mathcal{T}_{\varepsilon, \lambda}) \text{ to } (L^0(\mathcal{F}, K), \mathcal{T}_c)\}$ ,

where  $(E, \mathcal{T}_{\varepsilon, \lambda})$  is regarded as a topological module over the topological ring  $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})$ .

$E_{\varepsilon, \lambda}^*$  was first introduced in [3] and deeply developed in [5, 6]. Companying the locally  $L^0$ -convex topology  $E_c^*$  first occurred in [4] in an anonymous way, but one of the recent results in [1] showed that there had been another equivalent formulation of  $E_c^*$  before the locally  $L^0$ -convex topology occurred, namely  $E_I^*$  (see Definition 2.5 below). It is well known from [1] that  $E_c^* \subset E_{\varepsilon, \lambda}^*$  generally, and  $E_c^* = E_{\varepsilon, \lambda}^*$  if  $\mathcal{P}$  has the countable concatenation property (see Section 2).

With the above preparations, we present the main results of this paper as follows:

**Theorem 1.1** Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$ . Then  $E_{max}^* = E_{\varepsilon, \lambda}^*$ .

**Theorem 1.2** Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$  and  $f \in E_{min}^*$ . If  $(\Omega, \mathcal{F}, P)$  is a nonatomic probability space, then  $f(x) = 0, \forall x \in E$ .

From the above two theorems, it is easy to see that among the four kinds of random conjugate spaces only  $E_{\varepsilon, \lambda}^*$  and  $E_c^*$  are universally suitable for the current development of the theory of random conjugate spaces.

The remainder of this paper is organized as follows: in Section 2 we will briefly collect some necessary known facts and in Section 3 we will prove our main results.

## 2 Preliminaries

Denote by  $\bar{L}^0(\mathcal{F}, R)$  the set of all equivalence classes of extended  $R$ -valued measurable functions on  $(\Omega, \mathcal{F}, P)$ . Then it is well known from [7] that  $\bar{L}^0(\mathcal{F}, R)$  is a complete lattice under the ordering  $\leq$ :  $\xi \leq \eta$  iff  $\xi^0(\omega) \leq \eta^0(\omega)$ , for almost all  $\omega$  in  $\Omega$  (briefly, a.s.), where  $\xi^0$  and  $\eta^0$  are arbitrarily chosen representatives of  $\xi$  and  $\eta$ , respectively. Furthermore, every subset  $G$  of  $\bar{L}^0(\mathcal{F}, R)$  has a supremum, denoted by  $\bigvee G$ , and an infimum, denoted by  $\bigwedge G$ . Finally  $L^0(\mathcal{F}, R)$ ,

as a sublattice of  $\bar{L}^0(\mathcal{F}, R)$ , is also a complete lattice in the sense that every subset with upper bound has a supremum. The pleasant properties of  $\bar{L}^0(\mathcal{F}, R)$  are summarized as follows:

**Proposition 2.1** ([7]) For every subset  $G$  of  $\bar{L}^0(\mathcal{F}, R)$  there exist countable subsets  $\{a_n \mid n \in N\}$  and  $\{b_n \mid n \in N\}$  of  $G$  such that  $\bigvee G = \bigvee_{n \geq 1} a_n$  and  $\bigwedge G = \bigwedge_{n \geq 1} b_n$ . Further, if  $G$  is directed (dually directed) with respect to  $\leq$ , then the above  $\{a_n \mid n \in N\}$  (accordingly,  $\{b_n \mid n \in N\}$ ) can be chosen as nondecreasing (correspondingly, nonincreasing) with respect to  $\leq$ .

Specially,  $L_+^0 = \{\xi \in L^0(\mathcal{F}, R) \mid \xi \geq 0\}$ ,  $L_{++}^0 = \{\xi \in L^0(\mathcal{F}, R) \mid \xi > 0 \text{ on } \Omega\}$ , where for  $A \in \mathcal{F}$ , “ $\xi > \eta$ ” on  $A$  means  $\xi^0(\omega) > \eta^0(\omega)$  a.s. on  $A$  for any chosen representatives  $\xi^0$  and  $\eta^0$  of  $\xi$  and  $\eta$ , respectively. As usual,  $\xi > \eta$  means  $\xi \geq \eta$  and  $\xi \neq \eta$ . For an arbitrarily chosen representative  $\xi^0$  of  $\xi \in L^0(\mathcal{F}, K)$ , define the two  $\mathcal{F}$ -measurable random variables  $(\xi^0)^{-1}$  and  $|\xi^0|$  by  $(\xi^0)^{-1}(\omega) = 1/\xi^0(\omega)$  if  $\xi^0(\omega) \neq 0$ , and  $(\xi^0)^{-1}(\omega) = 0$  otherwise, and by  $|\xi^0|(\omega) = |\xi^0(\omega)|$ ,  $\forall \omega \in \Omega$ . Then the equivalent class  $Q(\xi)$  of  $(\xi^0)^{-1}$  is called the generalized inverse of  $\xi$ ; the equivalent class  $|\xi|$  of  $|\xi^0|$  is called the absolute value of  $\xi$ .

For any  $A \in \mathcal{F}$ ,  $A^c$  denotes the complement of  $A$ ,  $\tilde{A} = \{B \in \mathcal{F} \mid P(A \Delta B) = 0\}$  denotes the equivalence class of  $A$ , where  $\Delta$  is the symmetric difference operation,  $I_A$  the characteristic function of  $A$ , and  $\tilde{I}_A$  is used to denote the equivalence class of  $I_A$ ; given two  $\xi$  and  $\eta$  in  $L^0(\mathcal{F}, R)$ , and  $A = \{\omega \in \Omega \mid \xi^0 \neq \eta^0\}$ , where  $\xi^0$  and  $\eta^0$  are arbitrarily chosen representatives of  $\xi$  and  $\eta$  respectively, then we always write  $[\xi \neq \eta]$  for the equivalence class of  $A$  and  $I_{[\xi \neq \eta]}$  for  $\tilde{I}_A$ , one can also understand the implication of such notations as  $I_{[\xi \leq \eta]}$ ,  $I_{[\xi < \eta]}$  and  $I_{[\xi = \eta]}$ .

**Definition 2.1** ([3, 8]) (1) Let  $E$  be a linear space over  $K$ , then a mapping  $f : E \rightarrow L^0(\mathcal{F}, K)$  is called a random linear functional on  $E$  if  $f$  is linear;

(2) If  $E$  is a linear space over  $R$ , then a mapping  $f : E \rightarrow L^0(\mathcal{F}, R)$  is called a random sublinear functional on  $E$  if  $f(\alpha x) = \alpha \cdot f(x)$  for any positive real number  $\alpha$  and  $x \in E$ , and  $f(x + y) \leq f(x) + f(y)$ ,  $\forall x, y \in E$ ;

(3) Let  $E$  be a linear space over  $K$ , then a mapping  $f : E \rightarrow L_+^0$  is called a random seminorm on  $E$  if  $f(\alpha x) = |\alpha| \cdot f(x)$ ,  $\forall \alpha \in K$  and  $x \in E$ , and  $f(x + y) \leq f(x) + f(y)$ ,  $\forall x, y \in E$ ;

(4) Let  $E$  be a left module over the algebra  $L^0(\mathcal{F}, K)$ , then a mapping  $f : E \rightarrow L^0(\mathcal{F}, K)$  is called a  $L^0$ -linear functional on  $E$  if  $f$  is a module homomorphism;

(5) Let  $E$  be a left module over the algebra  $L^0(\mathcal{F}, R)$ , a mapping  $f : E \rightarrow L^0(\mathcal{F}, R)$  is called an  $L^0$ -sublinear functional on  $E$  if  $f$  is a random sublinear functional on  $E$  such that  $f(\xi \cdot x) = \xi \cdot f(x)$ ,  $\forall \xi \in L_+^0$  and  $x \in E$ ;

(6) Let  $E$  be a left module over the algebra  $L^0(\mathcal{F}, K)$ , then a mapping  $f : E \rightarrow L_+^0$  is called an  $L^0$ -seminorm on  $E$  if  $f$  is a random seminorm on  $E$  such that  $f(\xi \cdot x) = |\xi| \cdot f(x)$ ,  $\forall \xi \in L^0(\mathcal{F}, K)$  and  $x \in E$ .

**Definition 2.2** ([1, 3]) An ordered pair  $(E, \mathcal{P})$  is called a random locally convex space over  $K$  with base  $(\Omega, \mathcal{F}, P)$  if the following three conditions are satisfied:

- (1)  $E$  is a linear space over  $K$ ;
- (2)  $\mathcal{P}$  is a family of random seminorms on  $E$  with base  $(\Omega, \mathcal{F}, P)$ ;
- (3)  $\bigvee \{\|x\| \mid \|\cdot\| \in \mathcal{P}\} = 0$  implies  $x = \theta$  (the null element of  $E$ ).

In addition, if  $E$  is a left module over the algebra  $L^0(\mathcal{F}, K)$  and each  $\|\cdot\|$  in  $\mathcal{P}$  is an  $L^0$ -seminorm then such a random locally convex space is called a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$ .

**Remark 1** Let  $(E, \mathcal{P})$  be a random locally convex space (a random locally convex module) over  $K$  with base  $(\Omega, \mathcal{F}, P)$ . If  $\mathcal{P}$  degenerates to a singleton  $\{\|\cdot\|\}$ , then  $(E, \|\cdot\|)$  is exactly a random normed space, briefly, an  $RN$  space (correspondingly, a random normed module,

briefly, an  $RN$  module). Specially,  $(L^0(\mathcal{F}, K), |\cdot|)$  is an  $RN$  module.

In the sequel, for a random locally convex space  $(E, \mathcal{P})$  with base  $(\Omega, \mathcal{F}, P)$  and for each finite subfamily  $\mathcal{Q}$  of  $\mathcal{P}$ ,  $\|\cdot\|_{\mathcal{Q}} : E \rightarrow L_+^0(\mathcal{F})$  always denotes the random seminorm of  $E$  defined by  $\|x\|_{\mathcal{Q}} = \bigvee \{\|x\| \mid \|\cdot\| \in \mathcal{Q}\}$ ,  $\forall x \in E$ , and  $\mathcal{F}(\mathcal{P})$  the set of finite subfamilies of  $\mathcal{P}$ .

For each random locally convex space  $(E, \mathcal{P})$  over  $K$  with base  $(\Omega, \mathcal{F}, P)$ ,  $\mathcal{P}$  can induce two kinds of topologies, namely the  $(\varepsilon, \lambda)$ -topology and the locally  $L^0$ -convex topology.

**Definition 2.3** ([1–3]) Let  $(E, \mathcal{P})$  be a random locally convex space over  $K$  with base  $(\Omega, \mathcal{F}, P)$ . For any positive real numbers  $\varepsilon$  and  $\lambda$  such that  $0 < \lambda < 1$ , and any  $\mathcal{Q} \in \mathcal{F}(\mathcal{P})$ , let  $N_{\theta}(\mathcal{Q}, \varepsilon, \lambda) = \{x \in E \mid P\{\omega \in \Omega \mid \|x\|_{\mathcal{Q}}(\omega) < \varepsilon\} > 1 - \lambda\}$ , then  $\{N_{\theta}(\mathcal{Q}, \varepsilon, \lambda) \mid \mathcal{Q} \in \mathcal{F}(\mathcal{P}), \varepsilon > 0, 0 < \lambda < 1\}$  is easily verified to be a local base at the null vector  $\theta$  of some Hausdorff linear topology. The linear topology is called the  $(\varepsilon, \lambda)$ -topology for  $E$  induced by  $\mathcal{P}$ .

From now on, the  $(\varepsilon, \lambda)$ -topology for each random locally convex space is always denoted by  $\mathcal{T}_{\varepsilon, \lambda}$  when no confusion occurs.

**Proposition 2.2** ([1–3]) Let  $(E, \mathcal{P})$  be a random locally convex space over  $K$  with base  $(\Omega, \mathcal{F}, P)$ . Then we have the following statements:

(1) The  $(\varepsilon, \lambda)$ -topology for  $L^0(\mathcal{F}, K)$  is exactly the topology of convergence in probability  $P$ , and  $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})$  is a topological algebra over  $K$ ;

(2) If  $(E, \mathcal{P})$  is a random locally convex module, then  $(E, \mathcal{T}_{\varepsilon, \lambda})$  is a topological module over the topological algebra  $L^0(\mathcal{F}, K)$ ;

(3) A net  $\{x_{\delta}, \delta \in \Gamma\}$  converges in the  $(\varepsilon, \lambda)$ -topology to some  $x$  in  $E$  iff for each  $\|\cdot\| \in \mathcal{P}$   $\{\|x_{\delta} - x\|, \delta \in \Gamma\}$  converges in probability  $P$  to 0.

The following locally  $L^0$ -convex topology is easily seen to be much stronger than the  $(\varepsilon, \lambda)$ -topology, and was first introduced by Filipović, Kupper and Vogelpoth in [4] for random locally convex modules.

**Definition 2.4** ([4]) Let  $(E, \mathcal{P})$  be a random locally convex space over  $K$  with base  $(\Omega, \mathcal{F}, P)$ . For any  $\mathcal{Q} \in \mathcal{F}(\mathcal{P})$  and  $\varepsilon \in L_{++}^0$ , let  $N_{\theta}(\mathcal{Q}, \varepsilon) = \{x \in E \mid \|x\|_{\mathcal{Q}} \leq \varepsilon\}$ . A subset  $G$  of  $E$  is called  $\mathcal{T}_c$ -open if for each  $x \in G$  there exists some  $N_{\theta}(\mathcal{Q}, \varepsilon)$  such that  $x + N_{\theta}(\mathcal{Q}, \varepsilon) \subset G$ ,  $\mathcal{T}_c$  denotes the family of  $\mathcal{T}_c$ -open subsets of  $E$ . Then it is easy to see that  $(E, \mathcal{T}_c)$  is a Hausdorff topological group with respect to the addition on  $E$ .  $\mathcal{T}_c$  is called the locally  $L^0$ -convex topology for  $E$  induced by  $\mathcal{P}$ .

From now on, the locally  $L^0$ -convex topology for each random locally convex space is always denoted by  $\mathcal{T}_c$  when no confusion occurs.

**Proposition 2.3** ([4]) Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$ . Then

(1)  $L^0(\mathcal{F}, K)$  is a topological ring endowed with its locally  $L^0$ -convex topology;

(2)  $E$  is a topological module over the topological ring  $L^0(\mathcal{F}, K)$  when  $E$  and  $L^0(\mathcal{F}, K)$  are endowed with their respective locally  $L^0$ -convex topologies;

(3) A net  $\{x_{\alpha} \mid \alpha \in \Gamma\}$  in  $E$  converges in the locally  $L^0$ -convex topology to  $x \in E$  iff  $\{\|x_{\alpha} - x\| \mid \alpha \in \Gamma\}$  converges in the locally  $L^0$ -convex topology of  $L^0(\mathcal{F}, K)$  to 0 for each  $\|\cdot\| \in \mathcal{P}$ .

$\mathcal{T}_c$  is called locally  $L^0$ -convex because it has a striking local base  $\mathcal{U}_{\theta} = \{B_{\mathcal{Q}}(\varepsilon) \mid \mathcal{Q} \subset \mathcal{P} \text{ finite and } \varepsilon \in L_{++}^0\}$ , each member  $U$  of which is:

- (i)  $L^0$ -convex:  $\xi \cdot x + (1 - \xi) \cdot y \in U$  for any  $x, y \in U$  and  $\xi \in L_+^0$  such that  $0 \leq \xi \leq 1$ ;
- (ii)  $L^0$ -absorbent: there is  $\xi \in L_{++}^0$  for each  $x \in E$  such that  $x \in \xi \cdot U$ ;
- (iii)  $L^0$ -balanced:  $\xi \cdot x \in U$  for any  $x \in U$  and any  $\xi \in L^0(\mathcal{F}, K)$  such that  $|\xi| \leq 1$ .

**Remark 2** Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$  endowed with the locally  $L^0$ -convex topology  $\mathcal{T}_c$ . Although  $E$  is a linear space,  $(E, \mathcal{T}_c)$  may not be a topological linear space since the scalar multiplication is not necessarily continuous, see [4] for details.

Historically, the earliest two notions of a random conjugate space of a random locally convex space were introduced in [3] and [9], respectively. As shown in [1, 2], it turned out that they just correspond to the  $(\varepsilon, \lambda)$ -topology and the locally  $L^0$ -convex topology in the context of a random locally convex module, respectively!

**Definition 2.5** ([9]) Let  $(E, \mathcal{P})$  be a random locally convex space over  $K$  with base  $(\Omega, \mathcal{F}, P)$ . A random linear functional  $f : E \rightarrow L^0(\mathcal{F}, K)$  is called an a.s. bounded random linear functional of type I if there are some  $\xi \in L^0_+$  and  $\mathcal{Q} \in \mathcal{F}(\mathcal{P})$  such that  $|f(x)| \leq \xi \cdot \|x\|_{\mathcal{Q}}, \forall x \in E$ . Denote by  $E_I^*$  the set of a.s. bounded random linear functional of type I on  $E$ . The module multiplication operation  $\cdot : L^0(\mathcal{F}, K) \times E_I^* \rightarrow E_I^*$  is defined by  $(\xi f)(x) = \xi(f(x)), \forall \xi \in L^0(\mathcal{F}, K), f \in E_I^*$  and  $x \in E$ . It is easy to see that  $E_I^*$  is a left module over  $L^0(\mathcal{F}, K)$ , called the random conjugate space of type I of  $E$ .

**Definition 2.6** ([3, 8]) Let  $(E, \mathcal{P})$  be a random locally convex space over  $K$  with base  $(\Omega, \mathcal{F}, P)$ . A random linear functional  $f : E \rightarrow L^0(\mathcal{F}, K)$  is called an a.s. bounded random linear functional of type II on  $E$  if there exist a countable partition  $\{A_i \mid i \in N\}$  of  $\Omega$  to  $\mathcal{F}$ , a sequence  $\{\xi_i \mid i \in N\}$  in  $L^0_+$  and a sequence  $\{\mathcal{Q}_i \mid i \in N\}$  in  $\mathcal{F}(\mathcal{P})$  such that  $|f(x)| \leq \sum_{i=1}^{\infty} \tilde{I}_{A_i} \cdot \xi_i \cdot \|x\|_{\mathcal{Q}_i}, \forall x \in E$ . Denote by  $E_{II}^*$  the  $L^0(\mathcal{F}, K)$ -module of a.s. bounded random linear functional of type II on  $E$ , called the random conjugate space of type II of  $E$ .

Propositions 2.12 and 2.13 below give the topological characterizations of an element in  $E_I^*$  and  $E_{II}^*$ , respectively.

**Proposition 2.4** ([1–3]) Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$  and  $f : E \rightarrow L^0(\mathcal{F}, K)$  a random linear functional. Then  $f \in E_I^*$  iff  $f$  is a continuous module homomorphism from  $(E, \mathcal{T}_c)$  to  $(L^0(\mathcal{F}, K), \mathcal{T}_c)$ , namely  $E_I^* = E_c^*$ .

**Proposition 2.5** ([1, 6]) Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$  and  $f : E \rightarrow L^0(\mathcal{F}, K)$  a random linear functional. Then  $f \in E_{II}^*$  iff  $f$  is a continuous module homomorphism from  $(E, \mathcal{T}_{\varepsilon, \lambda})$  to  $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})$ , namely  $E_{II}^* = E_{\varepsilon, \lambda}^*$ .

**Definition 2.7** ([1, 4]) Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$ .  $\mathcal{P}$  is called having the countable concatenation property if  $\sum_{n=1}^{\infty} \tilde{I}_{A_n} \cdot \|x\|_{\mathcal{Q}_n}$  still belongs to  $\mathcal{P}$  for any countable partition  $\{A_n \mid n \in N\}$  of  $\Omega$  to  $\mathcal{F}$  and any sequence  $\{\mathcal{Q}_n \mid n \in N\}$  in  $\mathcal{F}(\mathcal{P})$ .

**Proposition 2.6** ([1]) Let  $(E, \mathcal{P})$  be a random locally convex module. Then  $E_{\varepsilon, \lambda}^* = E_c^*$  if  $\mathcal{P}$  has the countable concatenation property (generally, it is obvious that  $E_c^* \subset E_{\varepsilon, \lambda}^*$ ). In particular  $E_{\varepsilon, \lambda}^* = E_c^*$  for any RN module  $(E, \|\cdot\|)$ .

### 3 Proofs of the Main Results

**Lemma 3.1** ([6]) Suppose  $E$  is a left module over the algebra  $L^0(\mathcal{F}, K)$ ,  $f : E \rightarrow L^0(\mathcal{F}, K)$  and  $\|\cdot\| : E \rightarrow L^0_+$  are such that  $|f(\xi \cdot x)| = \xi \cdot |f(x)|$  and  $\|\xi \cdot x\| = \xi \cdot \|x\|, \forall \xi \in L^0_+$  and  $x \in E$ . Denote  $B_{\|\cdot\|}(1) = \{x \in E \mid \|x\| \leq 1\}$ , then there exists  $\eta \in L^0_+$  such that  $|f(x)| \leq \eta \cdot \|x\|, \forall x \in E$  iff  $\bigvee \{|f(x)| \mid x \in B_{\|\cdot\|}(1)\} \in L^0_+$ .

We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** Since  $(L^0(\mathcal{F}, K), |\cdot|)$  is endowed with the  $(\varepsilon, \lambda)$ -topology,  $\{V_0(\varepsilon, \lambda) \mid \varepsilon$

and  $\lambda$  are real numbers,  $\varepsilon > 0$  and  $0 < \lambda < 1$  is a local base at 0 of the  $(\varepsilon, \lambda)$ -topology for  $L^0(\mathcal{F}, K)$ , where  $V_0(\varepsilon, \lambda) = \{\xi \in L^0(\mathcal{F}, K) \mid P(\{\omega \in \Omega \mid |\xi|(\omega) < \varepsilon\}) > 1 - \lambda\}$ . Select  $\{\varepsilon_n \mid n \in N\}$  and  $\{\lambda_n \mid n \in N\}$  to be any two sequences of positive numbers which both tend to 0 in a decreasing way and  $\lambda_n < 1$  for each  $n \in N$ , then clearly  $\{V_0(\varepsilon_n, \lambda_n) \mid n \in N\}$  is also a local base at 0 of the  $(\varepsilon, \lambda)$ -topology for  $L^0(\mathcal{F}, K)$ .

Since  $f : (E, \mathcal{T}_c) \rightarrow (L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})$  is continuous, for each  $n \in N$  there exist at least some  $\bar{\varepsilon}_n \in L^0_{++}$  and  $\|\cdot\|_n \in \mathcal{F}(\mathcal{P})$  such that

$$f(B_n(\bar{\varepsilon}_n)) \subset V_0(\varepsilon_n, \lambda_n),$$

where  $B_n(\bar{\varepsilon}_n) = \{x \in E \mid \|x\|_n \leq \bar{\varepsilon}_n\}$ . Let  $\|\cdot\|_{\mathcal{Q}_1} = \|\cdot\|_1$ ,  $\|\cdot\|_{\mathcal{Q}_n} = \|\cdot\|_1 \vee \cdots \vee \|\cdot\|_n, \forall n \geq 2$  and  $B_{\mathcal{Q}_n}(\bar{\varepsilon}_n) = \{x \in E \mid \|x\|_{\mathcal{Q}_n} \leq \bar{\varepsilon}_n\}$ , then

$$f(B_{\mathcal{Q}_n}(\bar{\varepsilon}_n)) \subset V_0(\varepsilon_n, \lambda_n)$$

and

$$B_{\mathcal{Q}_{n+1}}(1) \subset B_{\mathcal{Q}_n}(1), \forall n \in N,$$

where  $B_{\mathcal{Q}_n}(1) = \{x \in E \mid \|x\|_{\mathcal{Q}_n} \leq 1\}$ ,  $\forall n \in N$ .

Denote  $\{|f(x)| \mid x \in B_{\mathcal{Q}_n}(\bar{\varepsilon}_n)\}$  by  $G_n$  and  $\bigvee G_n$  by  $\eta_n$ . First, it is easy to see that  $G_n$  is a directed set in  $L^0_+$ , then there exists a sequence  $\{x_{n,k} \mid k \in N\}$  in  $B_{\mathcal{Q}_n}(\bar{\varepsilon}_n)$  such that

$$\{|f(x_{n,k})| \mid k \in N\} \nearrow \eta_n.$$

Since  $P(\{\omega \in \Omega \mid |f(x_{n,k})|(\omega) < \varepsilon_n\}) > 1 - \lambda_n, \forall k \in N$ , then

$$P(\{\omega \in \Omega \mid \eta_n(\omega) \leq 2\varepsilon_n\}) > 1 - \lambda_n, \forall n \in N.$$

From  $\eta_n = \bigvee G_n = \bigvee \{|f(x)| \mid x \in B_{\mathcal{Q}_n}(\bar{\varepsilon}_n)\}$ , we can obtain that

$$\frac{1}{\bar{\varepsilon}_n} \cdot \eta_n = \bigvee \{|f(x)| \mid x \in B_{\mathcal{Q}_n}(1)\}, \forall n \in N$$

and

$$\begin{aligned} P(\{\omega \in \Omega \mid \frac{1}{\bar{\varepsilon}_n(\omega)} \cdot \eta_n(\omega) < +\infty\}) &= P(\{\omega \in \Omega \mid \eta_n(\omega) < +\infty\}) \\ &\geq P(\{\omega \in \Omega \mid \eta_n(\omega) \leq 2\varepsilon_n\}) \geq 1 - \lambda_n. \end{aligned}$$

Since  $B_{\mathcal{Q}_{n+1}}(1) \subset B_{\mathcal{Q}_n}(1)$  and  $\lambda_n \searrow 0$ , it is clear that

$$P(\{\omega \in \Omega \mid \eta_n(\omega) < +\infty\}) \leq P(\{\omega \in \Omega \mid \eta_{n+1}(\omega) < +\infty\}),$$

$$\{P(\{\omega \in \Omega \mid \eta_n(\omega) < +\infty\}) \mid n \in N\} \nearrow 1$$

and

$$\tilde{\Omega} = \bigcup_{n=1}^{\infty} [\eta_n < \infty].$$

Taking  $A_1 = \{\omega \in \Omega \mid \frac{1}{\bar{\varepsilon}_1(\omega)} \cdot \eta_1^0(\omega) < +\infty\}$ ,  $A_i = \{\omega \in \Omega \mid \frac{1}{\bar{\varepsilon}_i^0(\omega)} \cdot \eta_i^0(\omega) < \frac{1}{\bar{\varepsilon}_{i-1}^0(\omega)} \cdot \eta_{i-1}^0(\omega) = +\infty\}$ ,  $\forall i \geq 2$ , then  $\{A_n \mid n \in N\}$  forms a countable partition of  $\Omega$  to  $\mathcal{F}$ . For each  $n \in N$ , define  $P_n : A_n \cap \mathcal{F} \rightarrow [0, 1]$  by  $P_n(A_n \cap F) = \frac{P(A_n \cap F)}{P(A_n)}, \forall F \in \mathcal{F}$ ,  $E^{(n)} = \tilde{I}_{A_n} \cdot E$ ,  $\|\cdot\|^{(n)}$  is the limitation of  $\|\cdot\|_{\mathcal{Q}_n}$  to  $E^{(n)}$ , and  $f^{(n)}$  is the limitation of  $f$  to  $E^{(n)}$ , then  $E^{(n)}$  is a left module over the algebra  $L^0(A_n \cap \mathcal{F}, K)$ . Applying Lemma 3.1 to  $E^{(n)}$ ,  $f^{(n)}$  and  $\|\cdot\|_{\mathcal{Q}_n}$  yields

$$\tilde{I}_{A_n} |f(x)| \leq \frac{\eta_n}{\bar{\varepsilon}_n} \cdot \tilde{I}_{A_n} \cdot \|x\|_{\mathcal{Q}_n}, \forall n \in N.$$

Let  $\xi_n = \frac{\eta_n}{\varepsilon_n} \cdot \tilde{I}_{A_n}$ , then

$$|f(x)| \leq \sum_{n=1}^{\infty} \tilde{I}_{A_n} \cdot \xi_n \cdot \|x\|_{\mathcal{Q}_n},$$

$\forall x \in E$ . By Definition 2.5 it is obvious that  $E_{max}^* \subset E_{\varepsilon, \lambda}^*$ . Finally, since it is clear that  $E_{\varepsilon, \lambda}^* \subset E_{max}^*$ , then we can obtain that  $E_{max}^* = E_{\varepsilon, \lambda}^*$ .

This completes the proof of Theorem 1.1.

Before giving the proof of Theorem 1.2, we need the following topological terminology and several lemmas on totally disconnected spaces.

**Definition 3.1** ([10]) A topological space  $(X, \mathcal{T})$  is called totally disconnected if for any  $x \in X$  and any neighborhood  $U$  of  $x$ , there is  $V \subset U$  that is both  $\mathcal{T}$ -open and  $\mathcal{T}$ -closed such that

$$x \in V \subset U.$$

From Definition 3.1, we can easily obtain the following:

**Lemma 3.2** Let  $(X, \mathcal{T})$  be a Hausdorff and totally disconnected space. Then any nonempty  $A \subset X$  is a connected subset iff  $A$  is a single point set.

Now let us recall the notion of an atom: Let  $(\Omega, \mathcal{F}, P)$  be a probability space, a set  $A \in \mathcal{F}$  is called an atom (or,  $P$ -atom) if  $P(A) > 0$  and if  $B \in \mathcal{F}$ ,  $B \subset A$ , then either  $P(B) = 0$  or  $P(A \setminus B) = 0$ . It is clear that if  $A_1$  and  $A_2$  are atoms, then either  $P(A_1 \cap A_2) = 0$  or  $P(A_1 \Delta A_2) = 0$ . Also, it is easy to see that  $(\Omega, \mathcal{F}, P)$  essentially has at most countably many disjoint atoms. In this paper, we say that  $(\Omega, \mathcal{F}, P)$  is essentially purely  $P$ -atomic if there exists at most, a countable family  $\{A_n \mid n \in \mathbb{N}\}$  of disjoint atoms such that  $\Omega = \sum_{n=1}^{\infty} A_n$  and such that for each  $A \in \mathcal{F}$  there is  $B$  in the  $\sigma$ -algebra generated by the family  $\{A_n \mid n \in \mathbb{N}\}$  such that  $P(A \Delta B) = 0$ . A probability space  $(\Omega, \mathcal{F}, P)$  without any atoms is called nonatomic.

For a nonatomic probability space  $(\Omega, \mathcal{F}, P)$ , the following two lemmas are known and very important for the proof of Lemma 3.5 below.

**Lemma 3.3** ([11]) If  $(\Omega, \mathcal{F}, P)$  is a nonatomic probability space, then the range of  $P$  is the whole interval  $[0, 1]$ .

**Lemma 3.4** Let  $(\Omega, \mathcal{F}, P)$  be a nonatomic probability space and  $A \in \mathcal{F}$  with  $P(A) > 0$ . Then there is a countable partition  $\{A_n \mid n \in \mathbb{N}\}$  of  $A$  to  $\mathcal{F}$  such that  $P(A_n) = \frac{1}{2^n} P(A)$ .

**Lemma 3.5** Let  $(\Omega, \mathcal{F}, P)$  be a nonatomic probability space,  $(E, \|\cdot\|)$  an  $RN$  module over  $K$  with the base  $(\Omega, \mathcal{F}, P)$ ,  $B^\circ(1) = \{x \in E \mid \|x\| < 1 \text{ on } \Omega\}$  and  $M = \{x \in B^\circ(1) \mid \exists m_x \in R, 0 < m_x < 1 \text{ such that } \|x\| < m_x \text{ on } \Omega\}$ . Then  $M$  is an  $L^0$ -convex,  $\mathcal{T}_c$ -closed and  $\mathcal{T}_c$ -open subset of  $E$ .

*Proof.* If  $x_1, x_2 \in M$ , then according to the definition of  $M$  there are two positive real numbers  $m_{x_1} < 1$  and  $m_{x_2} < 1$  such that  $\|x_1\| < m_{x_1}$  and  $\|x_2\| < m_{x_2}$  on  $\Omega$ . It is easy to see that  $\|\xi \cdot x_1 + (1 - \xi) \cdot x_2\| \leq |\xi| \cdot \|x_1\| + |1 - \xi| \cdot \|x_2\| < \max(m_{x_1}, m_{x_2}) < 1$  on  $\Omega$ , where  $\xi \in L_+^0$  with  $0 \leq \xi \leq 1$ . Thus,  $M$  is  $L^0$ -convex.

Now, we prove that  $M$  is a  $\mathcal{T}_c$ -closed subset of  $E$ . We only need to check that  $E \setminus M$  is  $\mathcal{T}_c$ -open and this can proceed in the following two cases:

Case(1): when  $y_1 \in E \setminus M$  and  $y_1 \notin B^\circ(1)$ , then there is  $D_1 \in \mathcal{F}$  with  $P(D_1) > 0$  such that  $\|y_1\| \geq 1$  on  $D_1$ . By Lemma 3.4, there is a countable partition  $\{D_{1,n} \mid n \in \mathbb{N}\}$  of  $D_1$  to  $\mathcal{F}$  such that  $P(D_{1,n}) = \frac{1}{2^n} P(D_1)$ . Let

$$\varepsilon_1 = \tilde{I}_{D_1^c} + \sum_{n=1}^{\infty} \frac{1}{n} \cdot \tilde{I}_{D_{1,n}}$$

and

$$B(\varepsilon_1) = \{x \in E \mid \|x\| \leq \varepsilon_1\},$$

then  $B(\varepsilon_1)$  is a neighborhood of 0, and for any  $\tilde{y}_1 \in y_1 + B(\varepsilon_1)$  it is easy to see that

$$\|y_1\| - \|\tilde{y}_1\| \leq \|y_1 - \tilde{y}_1\| \leq \varepsilon_1$$

and

$$\|\tilde{y}_1\| \geq \|y_1\| - \varepsilon_1 \geq 1 - \frac{1}{n} \text{ on } D_{1,n},$$

namely  $P(\|\tilde{y}_1\| \geq 1 - \frac{1}{n}) \geq P(D_{1,n}) > 0, \forall n \in N$ . Consequently, we have that  $\tilde{y}_1 \notin M$  and  $y_1 + B(\varepsilon_1) \subset E \setminus M$ .

Case(2): when  $y_2 \in E \setminus M$  and  $y_2 \in B^\circ(1)$ , then  $\|y_2\| < 1$  on  $\Omega$  and  $P(\{\omega \in \Omega \mid \|y_2\|(\omega) > 1 - \frac{1}{n}\}) > 0$  for each  $n \in N$  by the definition of  $M$ . Let  $H_n = \{\omega \in \Omega \mid \|y_2\|^0(\omega) > 1 - \frac{1}{n}\}$ ,  $\forall n \in N$ ,  $D_{2,n} = H_n \setminus H_{n+1} = \{\omega \in \Omega \mid 1 - \frac{1}{n} < \|y_2\|^0(\omega) \leq 1 - \frac{1}{n+1}\}$ , where  $y_2^0$  is an arbitrarily chosen representative of  $y_2$ , then for any  $i, j \in N$  and  $i \neq j$ ,  $D_{2,i} \cap D_{2,j} = \emptyset$  and  $H_i = \sum_{n=i}^\infty D_{2,n}$ . Assume that there is  $k \in N$  such that  $P(D_{2,n}) = 0, \forall n \geq k$ , then it implies that  $P(H_k) = \sum_{n=k}^\infty P(D_{2,n}) = 0$ , which is impossible. Hence, there exists a subsequence  $\{n_k \mid k \in N\}$  of  $N$  such that  $P(D_{2,n_k}) > 0, \forall k \in N$  and we can suppose, without loss of generality,  $P(D_{2,n}) > 0$  for each  $n \in N$ . Let

$$D_2 = \sum_{n=1}^\infty D_{2,n},$$

$$\varepsilon_2 = \tilde{I}_{D_2^c} + \sum_{n=1}^\infty \frac{1}{n} \cdot \tilde{I}_{D_{2,n}}$$

and

$$B(\varepsilon_2) = \{x \in E \mid \|x\| \leq \varepsilon_2\},$$

then  $B(\varepsilon_2)$  is a neighborhood of 0, and for any  $\tilde{y}_2 \in y_2 + B(\varepsilon_2)$  it is obvious that

$$\|y_2\| - \|\tilde{y}_2\| \leq \|y_2 - \tilde{y}_2\| \leq \varepsilon_2$$

and

$$\|\tilde{y}_2\| \geq \|y_2\| - \varepsilon_2 \geq 1 - \frac{1}{n} - \frac{1}{n} = 1 - \frac{2}{n} \text{ on } D_{2,n},$$

namely  $P(\|\tilde{y}_2\| \geq 1 - \frac{2}{n}) \geq P(D_{2,n}) > 0, \forall n \in N$ . Consequently, we have that  $\tilde{y}_2 \notin M$  and  $y_2 + B(\varepsilon_2) \subset E \setminus M$ .

From the two cases above, it is clear that  $M$  is a  $\mathcal{T}_c$ -closed subset of  $E$ .

Finally, we prove that  $M$  is a  $\mathcal{T}_c$ -open subset of  $E$  as follows: if  $y \in M$ , then  $y \in B^\circ(1)$  and there is a positive real number  $m_y < 1$  such that  $\|y\| < m_y$  on  $\Omega$ . Let  $\varepsilon^0 : \Omega \rightarrow R$  be defined by  $\varepsilon^0(\omega) = \frac{1-m_y}{2}$  for all  $\omega \in \Omega$  and  $\varepsilon \in L_{++}^0$  the equivalence class of  $\varepsilon^0$ , then  $B(\varepsilon) = \{x \in E \mid \|x\| \leq \varepsilon\}$  is a neighborhood of 0, and for any  $\tilde{y} \in y + B(\varepsilon)$

$$\|\tilde{y}\| - \|y\| \leq \|y - \tilde{y}\| \leq \varepsilon$$

and

$$\|\tilde{y}\| \leq m_y + \frac{1-m_y}{2} = \frac{1+m_y}{2} < 1$$

on  $\Omega$ . Hence,  $y + B(\varepsilon) \subset M$  and  $M$  is a  $\mathcal{T}_c$ -open subset of  $E$ .

This completes the proof. □



**Lemma 3.6** Let  $(\Omega, \mathcal{F}, P)$  be a nonatomic probability space and  $(E, \|\cdot\|)$  a random normed module over  $K$  with the base  $(\Omega, \mathcal{F}, P)$ . Then  $(E, \mathcal{T}_c)$  is a Hausdorff and totally disconnected space.

*Proof.* We only need to check that  $(E, \mathcal{T}_c)$  is totally disconnected. Let  $M$  be the same one as in Lemma 3.5,  $\mathcal{U}_\theta = \{B_{\|\cdot\|}(\varepsilon) \mid \varepsilon \in L_{++}^0\}$ , where  $B_{\|\cdot\|}(\varepsilon) = \{x \in E \mid \|x\| \leq \varepsilon\}$ . Then  $M$  is both  $\mathcal{T}_c$ -closed and  $\mathcal{T}_c$ -open subset of  $E$  by Lemma 3.5. For each  $\varepsilon \in L_{++}^0$ , it is easy to see that  $M$  and  $\varepsilon \cdot M$  are homeomorphic and

$$0 \in \varepsilon \cdot M \subset B_{\|\cdot\|}(\varepsilon).$$

Hence, by Definition 2.4 and Definition 3.1, we have that  $(E, \mathcal{T}_c)$  is totally disconnected.

This completes the proof.  $\square$

We can now prove Theorem 1.2.

**Proof of Theorem 1.2.** Since  $f \in E_{min}^*$ ,  $(E, \mathcal{T}_{\varepsilon, \lambda})$  is connected and  $(L^0(\mathcal{F}, K), \mathcal{T}_c)$  is a Hausdorff and totally disconnected space by Lemma 3.6, it is clear that  $f(E)$  is a single point set of  $L^0(\mathcal{F}, K)$  by Lemma 3.2. Hence,  $f(x) = 0, \forall x \in E$ .

This completes the proof.

**Lemma 3.7** Let  $(\Omega, \mathcal{F}, P)$  be a essentially purely  $P$ -atomic probability space and  $(E, \mathcal{P})$  a random locally convex module over  $K$  with the base  $(\Omega, \mathcal{F}, P)$ . Then  $f \in E_{min}^*$  iff there are finite  $P$ -atoms  $\{A_i \mid 1 \leq i \leq n\}$ ,  $\{\xi_i \mid 1 \leq i \leq n\}$  in  $L_+^0$  and  $\{Q_i \mid 1 \leq i \leq n\}$  in  $\mathcal{F}(\mathcal{P})$  such that

$$|f(x)| \leq \sum_{i=1}^n \tilde{I}_{A_i} \cdot \xi_i \cdot \|x\|_{Q_i}, \forall x \in E.$$

*Proof.* (1) Necessity: since  $(\Omega, \mathcal{F}, P)$  is essentially purely  $P$ -atomic and  $E_{min}^* \subset E_{\varepsilon, \lambda}^*$ , there exist  $\{\xi_i \mid i \in N\} \subset L_+^0$  and  $\{Q_i \mid i \in N\} \subset \mathcal{F}(\mathcal{P})$  such that

$$|f(x)| \leq \sum_{i=1}^\infty \tilde{I}_{A_i} \cdot \xi_i \cdot \|x\|_{Q_i},$$

where  $\mathcal{F}$  is generated by at most countably many disjoint atoms  $\{A_i \mid i \in N\}$ .

We will prove the following claim: there is  $n \in N$  such that  $\xi_i = 0$  on  $A_i, \forall i \geq n$ . Otherwise, there exist  $\{i_k \mid k \in N\} \subset N$ ,  $\{x_k \in E \mid k \in N\}$  and  $\{Q_{i_k} \mid k \in N\} \subset \mathcal{F}(\mathcal{P})$  such that on  $A_{i_k}$

$$\|\tilde{I}_{A_{i_k}} \cdot x_k\|_{Q_{i_k}} > 0$$

and

$$\tilde{I}_{A_{i_k}} \cdot |f(x_k)| > 0.$$

Taking  $y_k = \tilde{I}_{A_{i_k}} \cdot Q(|f(x_k)|) \cdot x_k$ , since  $A_{i_k}$  is a  $P$ -atom of  $\mathcal{F}$  for each  $k \in N$  and  $P(\sum_{k=1}^\infty A_{i_k}) < 1$ , then  $\{P(A_{i_k}) \mid k \in N\} \rightarrow 0$  and  $\{y_k \mid k \in N\}$  converges to 0 under  $\mathcal{T}_{\varepsilon, \lambda}$ . But  $\{f(y_{x_k}) \mid k \in N\}$  does not converge to 0 under  $\mathcal{T}_c$  by  $|f(y_{x_k})| = 1$  on  $A_{i_k}$ , which is a contradiction to  $f \in E_{min}^*$ .

(2) Sufficiency: if there are finite  $P$ -atoms  $\{A_i \mid 1 \leq i \leq n\}$ ,  $\{\xi_i \mid 1 \leq i \leq n\}$  in  $L_+^0$  and  $\{Q_i \mid 1 \leq i \leq n\} \subset \mathcal{F}(\mathcal{P})$  such that

$$|f(x)| \leq \sum_{i=1}^n \tilde{I}_{A_i} \cdot \xi_i \cdot \|x\|_{Q_i},$$

$\forall x \in E$ . Taking  $A = \sum_{i=1}^n A_i$ ,  $E_A = \tilde{I}_A \cdot E$  and  $f_A =$  the limitation of  $f$  on  $E_A$ , since  $A \cap \mathcal{F}$  is a  $\sigma$ -algebra generated by finite  $P$ -atoms, the  $(\varepsilon, \lambda)$ -topology is equivalent to the locally  $L^0$ -convex topology for  $E_A$ . Hence  $f_A : (E_A, \mathcal{T}_{\varepsilon, \lambda}) \rightarrow (\tilde{I}_A \cdot L^0(\mathcal{F}, K), \mathcal{T}_c)$  is a continuous

homomorphism, where the base of both  $E_A$  and  $\tilde{I}_A \cdot L^0(\mathcal{F}, K)$  is taken to be  $(A, A \cap \mathcal{F}, P_A)$  with  $P_A(A \cap F) = \frac{P(A \cap F)}{P(A)}$ ,  $\forall F \in \mathcal{F}$ . Finally, since  $f = 0$  on  $\tilde{I}_{A^c} \cdot E$ , it is clear that  $f$  is a continuous homomorphism from  $(E, \mathcal{T}_{\varepsilon, \lambda})$  to  $(L^0(\mathcal{F}, K), \mathcal{T}_c)$  and  $f \in E_{min}^*$ .

This completes the proof.  $\square$

**Corollary 3.1** Let  $(E, \mathcal{P})$  be a random locally convex module over  $K$  with base  $(\Omega, \mathcal{F}, P)$  such that  $\mathcal{F}$  has at least a  $P$ -atom. Then  $f \in E_{min}^*$  iff there are finite  $P$ -atoms  $\{A_i \mid 1 \leq i \leq n\}$ ,  $\{\xi_i \mid 1 \leq i \leq n\}$  in  $L_+^0$  and  $\{\mathcal{Q}_i \mid 1 \leq i \leq n\}$  in  $\mathcal{F}(\mathcal{P})$  such that

$$|f(x)| \leq \sum_{i=1}^n \tilde{I}_{A_i} \cdot \xi_i \cdot \|x\|_{\mathcal{Q}_i},$$

$\forall x \in E$ .

*Proof.* It follows immediately from Theorem 1.2 and Lemma 3.7.

This completes the proof.  $\square$

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